Explicit Renormalization Group for $D = 2$ Random Bond Ising Model with Long-Range Correlated Disorder

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Abstract We investigate the explicit renormalization group for fermionic field theoretic representation of two-dimensional random bond Ising model with long-range correlated disorder. We show that a new fixed point appears by introducing a long-range correlated disorder. Such as the one has been observed in previous works for the bosonic (φ^4) description. We have calculated the correlation length exponent and the anomalous scaling dimension of fermionic fields at this fixed point. Our results are in agreement with the extended Harris criterion derived by Weinrib and Halperin.

1 Introduction

Critical properties of systems with short-range and long-range correlated randomness have been studied extensively [1–8]. One important question to address is whether the introduction of weak randomness changes the universality class of transition. That is, in the renormalization group (RG) language, whether the disorder is relevant at the critical point of pure system or not. According to the well known Harris criterion [3, 6], disorder is irrelevant if $d\nu > 2$, where d is the dimensionality and ν is the correlation length exponent of pure system. This criterion should be modified in the presence of long-range correlations in the disorder. A special type of such a disorder has been considered by Weinrib and Halperin. They showed that the disorder with power law correlation $\sim x^{2\rho - d}$ (for large separations x) is irrelevant if $[(d - 2\rho)\nu - 2] > 0$ for $\rho > 0$, whereas the usual Harris criterion recovers for $\rho < 0$ [7]. Therefore the existence of long-range correlations in the disorder would have significant effect in the sense that they can change the universality class of phase transitions.

The most useful method to study disordered systems in RG language is the Replica method. By employing this method, one can average out the free energy using a trick based on the equation $\ln Z = \lim_{n \to 0} \frac{Z^n - 1}{n}$. The idea is then to average Z^n. However, RG analysis

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can be implemented explicitly without averaging on disorder [14]. Using this method, one can avoid some of the mathematical problems e.g. the $n \to 0$ limit in the replica method. Moreover, application of the explicit method in some cases, such as the one studied in this letter, is more straightforward.

In this paper we consider random-bond Ising model with fermionic action and long-range correlated disorder. The effect of short-range correlated disorder has been studied in previous works [9–14]. Also the φ^4 version of this model is investigated through the double expansion near four dimensions with short-range and long-rang disorder [7, 8]. We found a new long-range fixed point and we calculate both correlation length exponent and scaling dimension of fermionic field at this fixed point.

2 Explicit Renormalization Group

Two dimensional Ising model near its critical point can be described by the massive free fermionic action

$$S = \frac{1}{2} \sum_x \bar{\psi}(x)(\hat{\partial} + m)\psi(x), \tag{1}$$

$\psi = \binom{\psi_1}{\psi_2}$ is a two component Grassmannian field ($\psi_i^\dagger = \psi_i$) and

$$\bar{\partial} = \sigma_3 \partial_1 + \sigma_1 \partial_2, \qquad \bar{\psi} = \psi^T i \sigma_2, \tag{2}$$

where σ_i are Pauli matrices. The two point function of the Grassmannian fields of the model in the momentum space is

$$G_0(p)(2\pi)^2 \delta(p+q) = \langle \psi(p)\psi(q) \rangle = \frac{-i\hat{p} + m}{p^2 + m^2}(2\pi)^2 \delta(p+q), \tag{3}$$

where $\psi(p)$ is the Fourier transform of $\psi(x)$

$$\psi(p) = \sum_x \psi(x) \exp(-ip \cdot x) \tag{4}$$

and

$$\psi(x) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \psi(p) \exp(ip \cdot x). \tag{5}$$

The randomness can be inserted into the action in the following way

$$S = \frac{1}{2} \sum_x \bar{\psi}(x)(\hat{\partial} + m + c(x))\psi(x), \tag{6}$$

where $c(x)$ is a random variable with the following correlation function in the momentum space

$$\begin{aligned} \langle c(p) \rangle &= 0, \\ \langle c(p)c(q) \rangle &= (D_0 + D_\rho |p|^{-2\rho})\delta(p+q). \end{aligned} \tag{7}$$

Here D_0 and D_ρ are short-range and long-range disorder strengths respectively. Positivity of two point function imposes some restrictions on D_0 and D_ρ for example D_ρ should be

positive for $\rho > 0$ and D_0 should be positive for $\rho < 0$. Here we consider the case with $0 < \rho < 1$.

We can write the action in the momentum space as

$$S = \frac{1}{2} \iint_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \bar\psi(-p)(i\hat p + m)\psi(p)$$

$$+ \frac{1}{2} \iiiint_{-\pi}^{\pi} \frac{d^2 p_1}{(2\pi)^2} \frac{d^2 p_2}{(2\pi)^2} \bar\psi(p_1) c(-p_1 - p_2)\psi(p_2), \qquad (8)$$

where

$$c(p) = \sum_x c(x)\exp(-ip.x).$$

It is convenient to introduce a diagrammatic representation. We have the following vertex

Wavy line stands for $\bar\psi$, solid line for ψ and zigzag represents the c insertion. Momentum conservation should be regarded at each vertex. So we have the following graphs for vertex renormalization

where the dashed line represents $\bar\psi\psi$ propagator. Symmetry factors will be canceled with $n!$ of perturbation expansion in each order. The key equation will be [14]

$$c'(r_1, r_2) = c(r_1 + r_2) - \int_q c(r_1 - q)G_0(q)c(q + r_2)$$

$$+ \int_{q_1} \int_{q_2} c(r_1 - q_1)G_0(q_1)c(q_1 - q_2)G_0(q_2)c(q_2 + r_2)$$

$$- \int_{q_1} \int_{q_2} \int_{q_3} c(r_1 - q_1)G_0(q_1)c(q_1 - q_2)G_0(q_2)c(q_2 - q_3)G_0(q_3)c(q_3 + r_2).$$

$$(9)$$

The random function in the original action has zero mean and we should keep the mean value fixed after RG transformation. So the mean value of new function should be extracted and in fact it will renormalize the mass and the kinetic term. Then we need to introduce a field renormalization constant Z to keep the coefficient of kinetic term equal to $\frac{1}{2}$, just as in the original action. After rescaling momentums ($r \to \lambda r$) and fields ($\psi \to \frac{\lambda^{3/2}}{\sqrt Z}\psi$), the renormalized random function and mass will be

$$c_r(p_1, p_2) = \frac{1}{\lambda Z}[c'(p_1/\lambda, p_2/\lambda) - \langle c'(p_1/\lambda, p_2/\lambda)\rangle], \qquad (10)$$

$$m_r = \frac{\lambda}{Z}(m + \langle c'\rangle_{r=0}). \qquad (11)$$

A simple expansion of the second term of (9) in powers of r, with $m = 0$, leads to the following expression for field renormalization constant to first order in disorder strength

$$Z = 1 + \frac{D_\rho}{4\pi^{1+2\rho}}(\lambda^{2\rho} - 1) + O(D^2). \tag{12}$$

The short-range correlated disorder does not contribute up to this order.

So the renormalized mass is

$$m_r = \lambda m \left\{ 1 - \frac{1}{2\pi}(\ln\lambda)D_0 - \frac{1+\rho}{4\rho\pi^{1+2\rho}}(\lambda^{2\rho} - 1)D_\rho + \left(\left(\frac{\ln\lambda}{2\pi}\right)^2 - I_1 \right)D_0^2 \right.$$

$$\left. - (I_2 + I_3 + I_5)D_0 D_\rho - (I_4 + I_6)D_\rho^2 + \cdots \right\}, \tag{13}$$

where the I_j's are

$$I_1 = \int_{q_1}\int_{q_2} \frac{-q_2.(q_2 - q_1) + m^2}{(q_1^2 + m^2)(q_2^2 + m^2)((q_2 - q_1)^2 + m^2)}, \quad |q_1 - q_2| \geq \frac{\pi}{\lambda},$$

$$I_2 = \int_{q_1}\int_{q_2} \frac{(-q_1.(q_2 - q_1) + m^2)|q_1|^{-2\rho}}{(q_1^2 + m^2)(q_2^2 + m^2)((q_2 - q_1)^2 + m^2)}, \quad |q_1 - q_2| \geq \frac{\pi}{\lambda},$$

$$I_3 = \int_{q_1}\int_{q_2} \frac{(-q_1.q_2 + m^2)|q_2 - q_1|^{-2\rho}}{(q_1^2 + m^2)(q_2^2 + m^2)((q_2 - q_1)^2 + m^2)}, \quad |q_1 - q_2| \geq \frac{\pi}{\lambda},$$

$$I_4 = \int_{q_1}\int_{q_2} \frac{(-q_1.(q_2 + q_1) + m^2)|q_1|^{-2\rho}|q_2 - q_1|^{-2\rho}}{(q_1^2 + m^2)(q_2^2 + m^2)((q_2 - q_1)^2 + m^2)}, \quad |q_1 - q_2| \geq \frac{\pi}{\lambda},$$

$$I_5 = \int_{q_1}\int_{q_2} \frac{(-q_1.(q_1 + 2q_2) + m^2)|q_2 - q_1|^{-2\rho}}{(q_1^2 + m^2)^2(q_2^2 + m^2)},$$

$$I_6 = \int_{q_1}\int_{q_2} \frac{(-q_1.(q_1 + 2q_2) + m^2)|q_2 - q_1|^{-2\rho}|q_1|^{-2\rho}}{(q_1^2 + m^2)^2(q_2^2 + m^2)}$$

we did not include the contribution of Z in the second order.

The renormalized values of disorder strengths can also be obtained by calculating the correlations of renormalized random function $c_r(p_1, p_2)$. Up to second order of bare strengths, the renormalized strengths are

$$D_{0r} = D_0 \left(1 - \frac{\ln\lambda}{2\pi}D_0 - \frac{1+\rho}{2\rho\pi^{1+2\rho}}(\lambda^{2\rho} - 1)D_\rho + \cdots \right), \tag{14}$$

$$D_{\rho r} = \lambda^{2\rho}D_\rho \left(1 - \frac{\ln\lambda}{2\pi}D_0 - \frac{1+\rho}{2\rho\pi^{1+2\rho}}(\lambda^{2\rho} - 1)D_\rho + \cdots \right). \tag{15}$$

By differentiating these equations with respect to $\ln\lambda$, and then replacing the bare parameters in terms of renormalized ones, one can obtain the following Wilson's functions

$$\frac{dm_r}{d\ln\lambda} = m_r \left(1 - \frac{1}{2\pi}D_{0r} - \frac{1+\rho}{2\pi^{1+2\rho}}D_{\rho r} + \cdots \right), \tag{16}$$

$$\frac{dD_{0r}}{d\ln\lambda} = -D_{0r} \left(\frac{1}{\pi}D_{0r} + \frac{1+\rho}{\pi^{1+2\rho}}D_{\rho r} + \cdots \right), \tag{17}$$

$$\frac{dD_{\rho r}}{d\ln\lambda} = D_{\rho r}\left(2\rho - \frac{1}{\pi}D_{0r} - \frac{1+\rho}{\pi^{1+2\rho}}D_{\rho r} + \cdots\right). \tag{18}$$

It is clear from the equations that at the Gaussian fixed point ($D_0^* = D_\rho^* = 0$), D_0 is marginally irrelevant and D_ρ is relevant. So, by introduction of small amount of long-range correlated disorder it becomes unstable in D_ρ direction. Apart from the trivial Gaussian fixed point, we see that there is a nontrivial fixed point at $D_0^* = 0$ and $D_\rho^* = \frac{2\rho}{1+\rho}\pi^{1+2\rho}$. The new fixed point is attractive in all directions in the D_0 and D_ρ plane. The RG flows starting in the vicinity of Gaussian fixed point, end up at the nontrivial fixed point. One of the most important critical exponents is the correlation length exponent, which turns out to be different at two fixed points, $\nu = 1$ at the Gaussian fixed point and $\nu = \frac{1}{1-\rho}$ at the nontrivial fixed point. The result is in agreement with [7]. A good numerical confirmation of this relation for the special choice of $\rho = \frac{1}{2}$ can be found in [15].

Also the field renormalization would acquire an anomalous dimension which is defined through the asymptotic behavior of vertex function in the long wavelength limit as

$$\Gamma^{(2)}(p) \sim p^{1-\eta} \tag{19}$$

so at the nontrivial fixed point we have

$$\eta = \frac{dZ}{d\ln\lambda} = \frac{\rho}{2\pi^{1+2\rho}}D_{\rho r}^* = \frac{\rho^2}{1+\rho}. \tag{20}$$

Now we want to show that, some less singular terms in the correlation functions of c_r, which were omitted in (7), are irrelevant. We start with a general form of the correlation function of disorder

$$\langle c(p)\rangle = 0,$$
$$\langle c(p)c(q)\rangle = \left(D_0 + \sum_{i=1}^{n} D_{\rho_i}|p|^{-2\rho_i}\right)\delta(p+q), \tag{21}$$

and then find the Wilson's functions for the D_0 and D_{ρ_i}

$$\frac{dD_{0r}}{d\ln\lambda} = -D_{0r}\left(\frac{1}{\pi}D_{0r} + \sum_{i=1}^{n}\frac{1+\rho_i}{\pi^{1+2\rho_i}}D_{\rho_i r} + \cdots\right), \tag{22}$$

$$\frac{dD_{\rho_i r}}{d\ln\lambda} = D_{\rho_i r}\left(2\rho_i - \frac{1}{\pi}D_{0r} - \sum_{j=1}^{n}\frac{1+\rho_j}{\pi^{1+2\rho_j}}D_{\rho_j r} + \cdots\right). \tag{23}$$

Here we have n nontrivial fixed points at $D_0^* = 0$ and all $D_{\rho_i}^* = 0$ except one of them, say $D_{\rho_l}^*$ which is $\frac{2\rho_l}{1+\rho_l}\pi^{1+2\rho_l}$. At these fixed points the RG eigenvalues are $-2\rho_l$ and $2(\rho_i - \rho_l)$ which means that there is just one stable fixed point (with $\rho_l > \rho_i$ for all i). Other fixed points are unstable at least in one direction.

Finally we want to compare the above results with the replica results. In the case of short-range disorder, as pointed out by Murthy [14], the results of the replica method and the explicit method are in complete agreement. The replica action for the model considered here can be easily obtained,

$$S = \frac{1}{2}\sum_{x,\alpha}\bar{\psi}(x)(\hat{\partial}+m)\psi(x) - \frac{D_0}{8}\sum_{x,\alpha,\beta}\bar{\psi}_\alpha(x)\psi_\alpha(x)\bar{\psi}_\beta(x)\psi_\beta(x)$$

$$- \frac{D_\rho}{8}\sum_{x,y,\alpha,\beta}\bar{\psi}_\alpha(x)\psi_\alpha(x)\frac{1}{|x-y|^{2-2\rho}}\bar{\psi}_\beta(y)\psi_\beta(y). \tag{24}$$

The third term in this action is nonlocal, and the computation of the β functions are not easily tractable. This shows the advantage of explicit calculations with which one can avoid such a nonlocal action. The other advantage of this method is that there is no restriction on the distribution function of disorder while it should have a Gaussian distribution in the replica method.

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